Gravity currents as a test case:

do simple mathematical models produce good physical insights,

or do good insights generate powerful models?

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Outline of presentation

- Introduction/Motivation
- Results for gravity currents
- Results for intrusions
- Concluding remarks

Note: we shall omit the mathematical details as much as possible.

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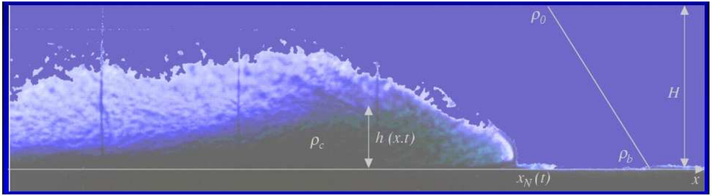

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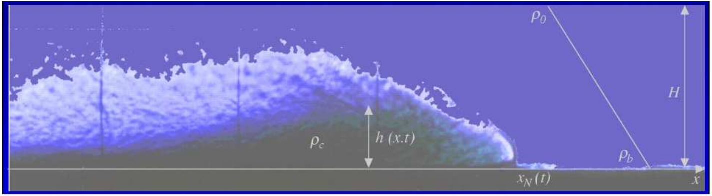

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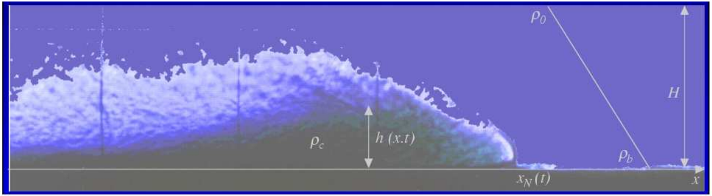

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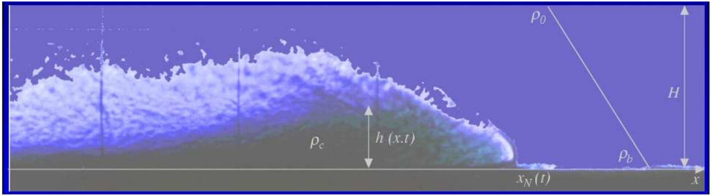

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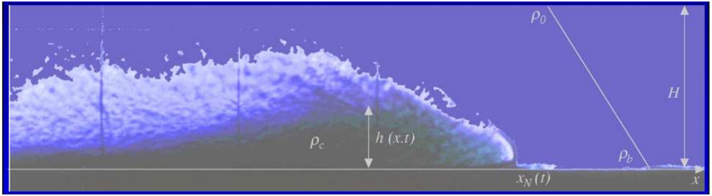

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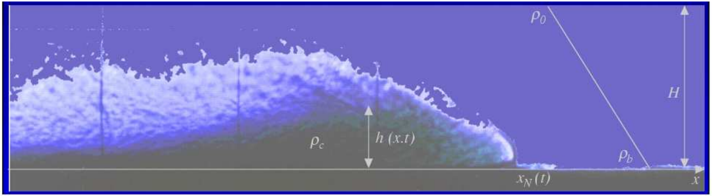

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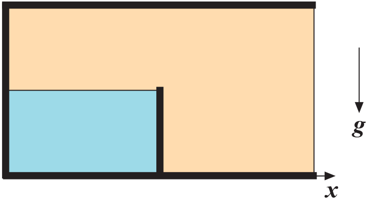
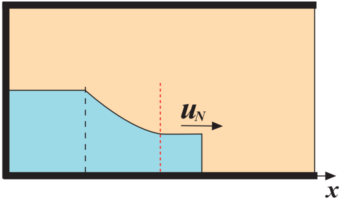
How can this be achieved? What do we learn? How good are the results?

Typical case: “dam break” of saltwater in freshwater



$t = 0$, with dam, no motion



$t > 0$, no dam, with motion

Density of ambient (yellow) is ρ_a. Density of current (blue) is ρ_c.

No motion appears when $\rho_c = \rho_a$. A gravity current appears when $\rho_c > \rho_a$.

First puzzle: the gravity g which acts in vertical direction (-z) drives a flow in the horizontal direction (x).

Due to g the pressure p is proportional to ρ and h (layer thickness) There is $\Delta p \propto (\rho_c - \rho_a)gh$ difference over Δx of the dam.

After removal of the dam, the $\partial p / \partial x$ drives x motion with speed u.

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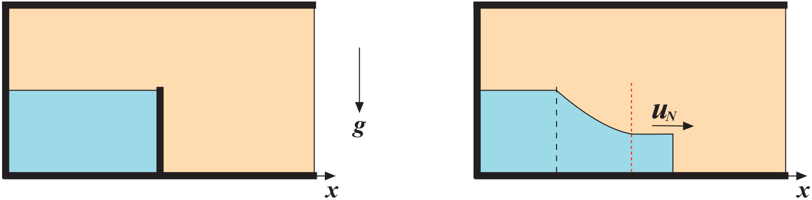

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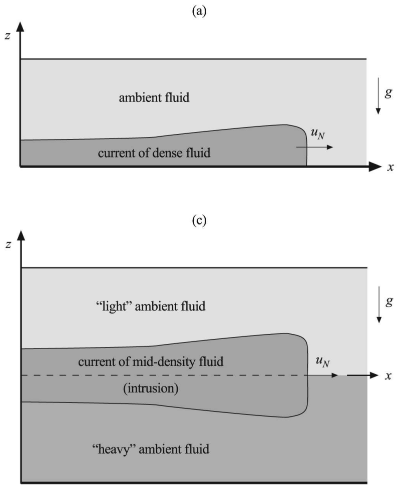
Typical configurations



Quick estimates

Driving effect: the "reduced gravity".

$$\text{Define} \quad \Delta\rho = \rho_c - \rho_a; \quad \varepsilon_a = \frac{\Delta\rho}{\rho_a}; \quad \varepsilon_c = \frac{\Delta\rho}{\rho_c};$$
$$g'_a = |\varepsilon_a| g; \quad g'_c = |\varepsilon_c| g;$$

and

$$g' = \frac{|\Delta\rho|}{\max(\rho_a, \rho_c)}\, g = \min(g'_a, g'_c).$$

Boussinesq system $|\varepsilon_c|, |\varepsilon_a| \ll 1$, allows the approximations

$$\rho_c = \rho_a, \quad \text{and} \quad g' = g'_a \approx g'_c$$

Lock height h_0. IF the mean reduced potential energy of a particle in the lock $(1/2)g'h_0\rho_a$ is converted to the kinetic energy $(1/2)U^2\rho_a$, then

$$\text{The typical speed is} \qquad U = (g'h_0)^{1/2}.$$

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The pressure/buoyancy driving is “balanced” by either inertial, or viscous, adjustment of the fluid.

The Reynolds number expresses Inertial/Viscous effects

$$Re = \frac{UL}{\nu} = \frac{\sqrt{g' h_0} h_0}{\nu}.$$

In most cases $Re \gg 1$.

Classification

- Constant (fixed)/non-constant volume.
- Inviscid/viscous.
- Boussinesq/non-Boussinesq.
- Homogeneous/stratified ambient.
- Gravity current/intrusion.
- Two-dimensional (2D) rectangular geometry/axisymmetric.
- Rotating/non-rotating frame (and ambient).
- Compositional/particle driven.

The gravity current is a very complex, multi-faced, and parameter-rich physical manifestation. Difficulties in the flow-field problem: two-fluid, time dependent, strong variations and instabilities (near the interface), different scales in x and z directions, etc.

here

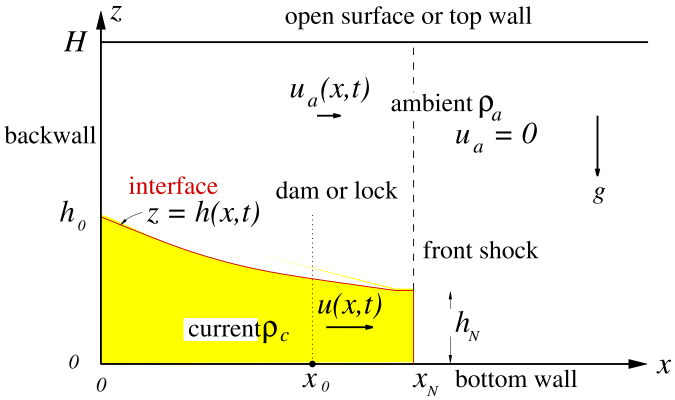
How can we proceed?

Suggestion: Let us use approximations and idealizations: sharp interface, thin layer, fully inviscid (or fully viscous), instantaneous release.

Pessimists say: Hopeless case. No good estimates of approximation errors are available. The error bounds add up to significant % of solution.

Optimists answer: Let us try and see. Errors may be smaller than the bounds, and sometimes cancel each other.

Shallow-water (SW) inviscid, Boussinesq model



More simplifications:

One layer analysis: there is no motion in the ambient.

“Thin” current $h_N/x_N \ll 1$. Since the current is thin, we are interested in the x (not z) changes.

The variables of interest are $h(x,t)$, $u(x,t)$ (averaged). Derive equations:

Continuity equation:

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} = 0.$$

Momentum z: $0 = -\partial p_i / \partial z - \rho_i g$. Principle: p is continuous at $z = h(x,t)$.

Fundamental result $$\frac{\partial p_c}{\partial x} = \Delta \rho g \frac{\partial h(x,t)}{\partial x}.$$

Momentum x: average the equation over $[0,h]$. Use previous $\frac{\partial p_c}{\partial x}$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\Delta \rho}{\rho_c} g \frac{\partial h}{\partial x}.$$

We obtained a hyperbolic set of 2 PDEs for $h(x,t)$ and $u(x,t)$.

Easy to solve, BUT do we have all the needed boundary conditions?

Initial h and u at $t=0$ ok. $u=0$ at $x=0$ ok.

What happens at $x = x_N(t)$? A discontinuity (shock) may appear.

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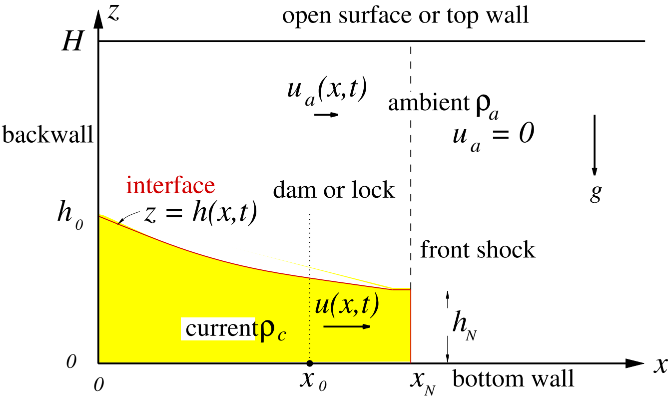
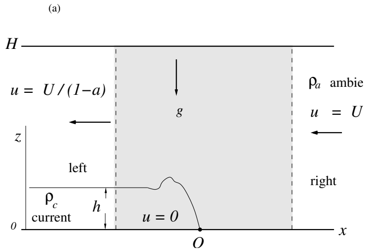
The SW current



Benjamin's (1968) classical result

For a *steady-state* long current, in a frame attached to the current.



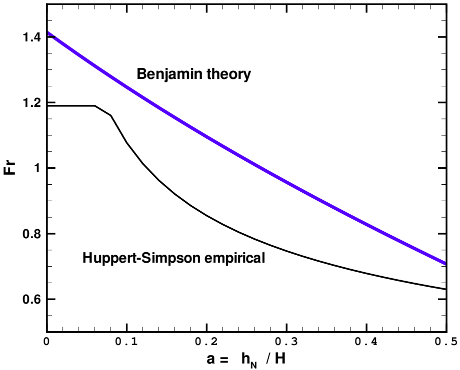
Use continuity of volume and flow force balance

$$\int_0^H (\rho u^2 + p)_l dz = \int_0^H (\rho u^2 + p)_r dz.$$

Result: $U = Fr(h/H) \cdot h^{1/2} \cdot [g']^{1/2}$

Fr is a simple function, of order 1 (called "Froude number.") here

Energy considerations give the restriction $h/H \leq 1/2$.



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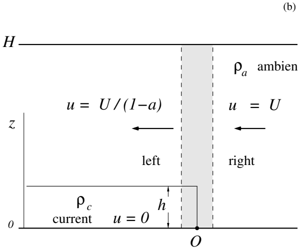
Insight: the previous *steady-state* analysis is relevant to the nose-shock of a *time-dependent* current



B. condition for the nose: $u_N = Fr(h_N/H) \cdot h_N^{1/2} \cdot [g']^{1/2}$

This closes the SW formulation.

SW formulation (2D)

Switch to dimensionless variables.

$$U = (g' h_0)^{1/2}; \quad T = \frac{x_0}{U}.$$

$$\{x^*, z^*, h^*, H^*, t^*, u^*, p^*\} = \{x_0 x, h_0 z, h_0 h, h_0 H, Tt, Uu, \rho_a U^2 p\}.$$

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The scaled (dimensionless) system is

$$\begin{bmatrix} h_t \\ u_t \end{bmatrix} + \begin{bmatrix} u & h \\ 1 & u \end{bmatrix} \begin{bmatrix} h_x \\ u_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The nose (front) condition needed at $x = x_N(t)$ is

$$u_N = h_N^{1/2}\, Fr(h_N/H), \quad \text{and} \quad \frac{h_N}{H} \le a_{\max} \approx 0.5,$$

where

$$Fr(h_N/H) = \begin{cases} 1.19 & (0 \le h_N/H < 0.075) \\ 0.5 H^{1/3} h_N^{-1/3} & (0.075 \le h_N/H \le 1). \end{cases}$$

See the simplification !

Recall: The full formulation of the problem is the Navier-Stokes equations

1. Continuity of volume
$$\nabla \cdot \mathbf{v} = 0;$$
2. Momentum balance
$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P - \rho g \hat{z} + \mu \nabla^2 \mathbf{v};$$
3. Density transport
$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = \kappa \nabla^2 \rho. \tag{2}$$

Here D/Dt is the “substantial” derivative.

Variables: $\mathbf{v}\{u, v, w\}$, P, and ρ functions of x, y, z, t.

A very difficult hyperbolic-parabolic PDEs problem.

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A very difficult hyperbolic-parabolic PDEs problem.

The SW model has only two variables, functions of x,t. Much simpler hyperbolic PDEs problem. But are the results useful? And accurate?

Immediate important results of SW formulation:

- The only free parameter is H = (height of lock)/(height of ambient).
- The initial propagation is with constant speed (2D case).
- A self-similar propagation develops eventually, spread $= t^{\beta}$.
  $\beta = 2/3$ (2D), $\beta = 1/2$ (axisym.).
- The transition position to “viscous” phase is predicted (for a given Reynolds number).
- The quantitative details are obtained within small computational effort (e.g., by finite-difference Lax-Wendroff method, in seconds).

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Comparisons to experiments and Navier-Stokes computations show: excellent qualitative agreement, fair quantitative agreement.

Axisymmetric example

Consider outward (radial) propagation in cylindrical geometry.

Consider two *very different* experiments:

(1) Hallworth, Ungarish and Huppert (2001). Circular tank of radius 13 m.

(2) Patterson, Simpson, Dalziel and van Heijst (2006). Wedge of 10°, length 2.35 m.

The fluids were salt- and fresh-water.

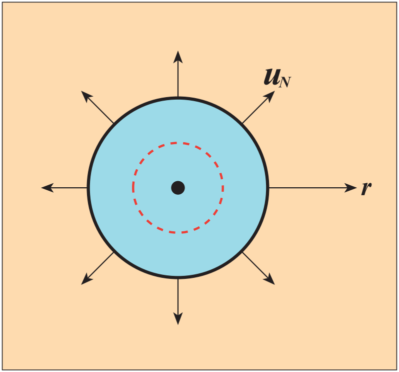
SW theory claims *All* are Boussinesq, large *Re* flows.

SW theory predicts: In scaled form both systems are *identical*. The only free parameter is H = (height of lock)/(height of ambient).

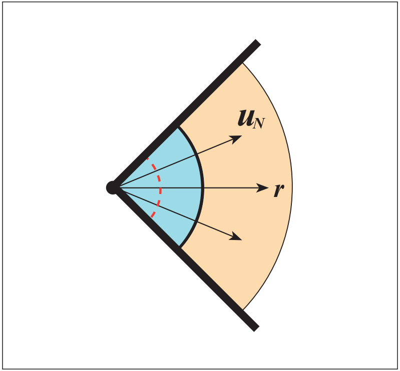
Scale: r with r_0; z with h_0.

u with $(g'h_0)^{1/2}$; t with $r_0/(g'h_0)^{1/2}$

Top view



Full cylinder



Wedge (Sector)

Conjecture: The flow variables are $h(r,t)$ and $u(r,t)$.

The experimental tank at Coriolis lab. Grenoble

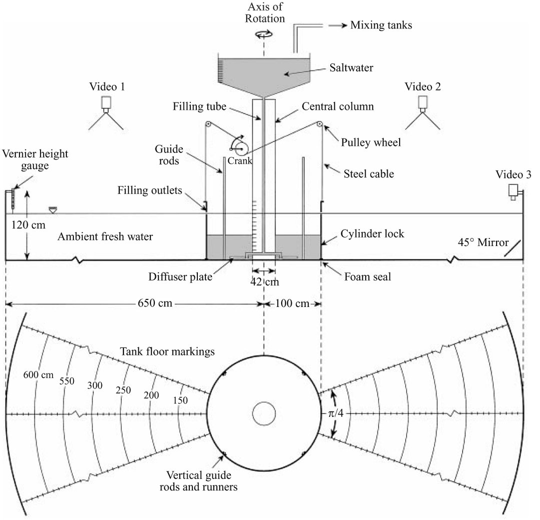
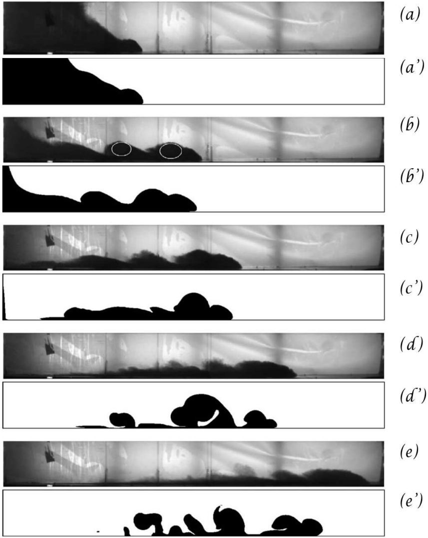


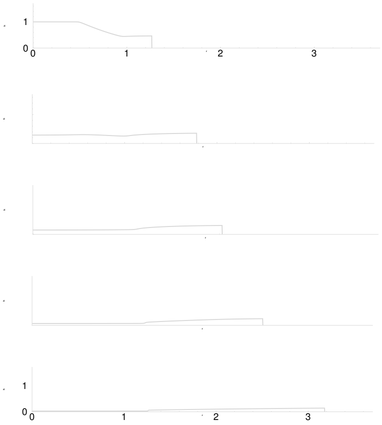
Table of compared experiments

Expt	r_0 cm	h_0 cm	H^* cm	H	g' $\mathrm{cm\,s^{-2}}$	Re $\times 10^4$	r_V	remark
S1	100	41.1	50.1	1.2	4.91	6	5.8	cylinder
S2	100	77.3	80.1	1.0	4.81	15	7.5	cylinder
S3	100	45.8	79.8	1.7	19.2	14	6.8	cylinder
S7	100	45.2	80.0	1.8	43.8	20	7.2	cylinder
P1	60	30	30	1.0	13.2	6	6.0	wedge
P2	60	22	30	1.4	13.2	4	5.4	wedge
P3	60	17.5	30	1.7	13.2	3	4.9	wedge
P4	60	9	30	3.3	13.2	1	3.7	wedge
P5	60	7.5	30	4.0	13.2	0.7	3.2	wedge

Wedge side view. $H = 1.7,\ t = 0.5, 1.5, 2, 3, 4.5$



SW results Ungarish 2007.



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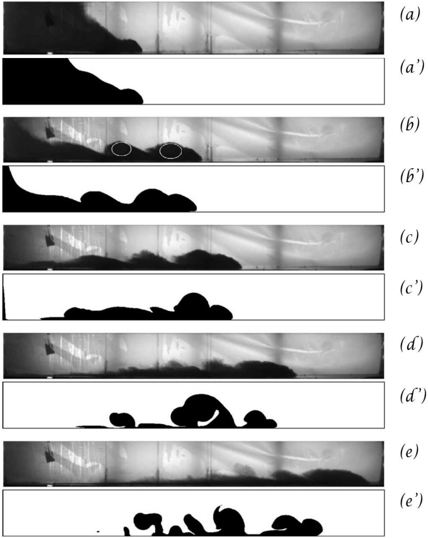

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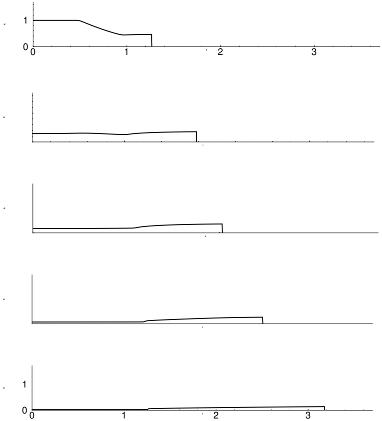

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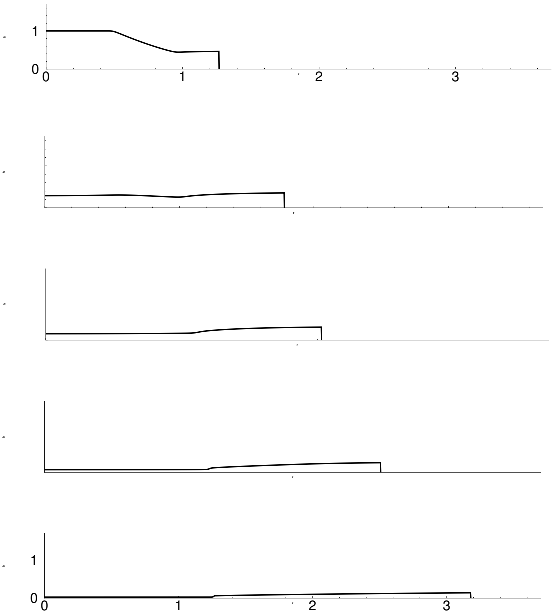

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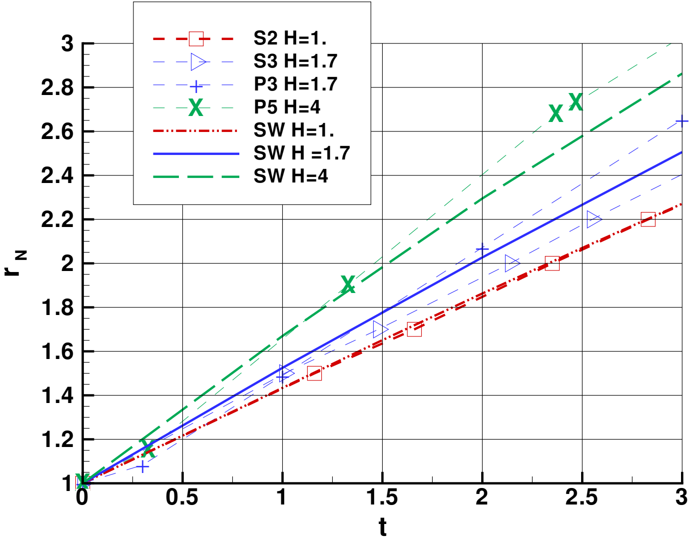
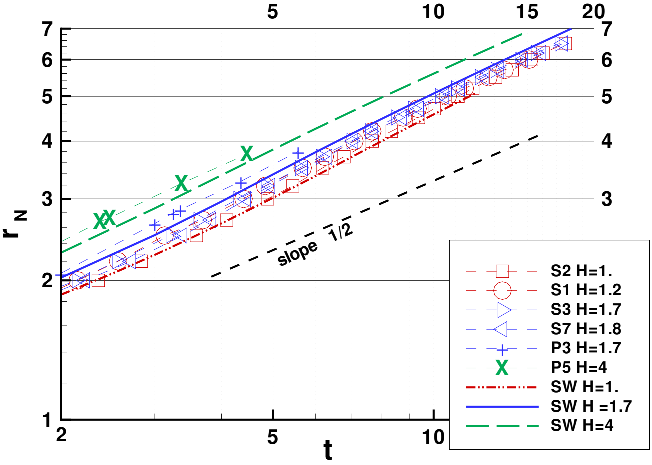
Experiments and theory, early times



Experiments and theory, later times

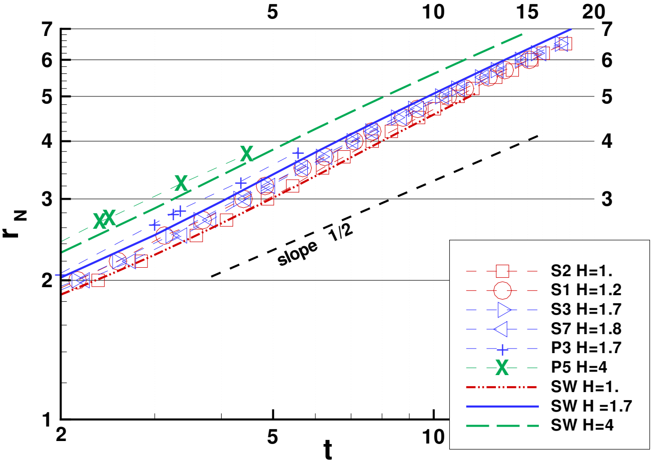


Conclusion: SW model predictions confirmed with good confidence.

The formulation covers a wide range of systems.

No adjustable constants were used.

Experiments and theory, later times



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Problem: Intrusions at mid-level of *stratified* container

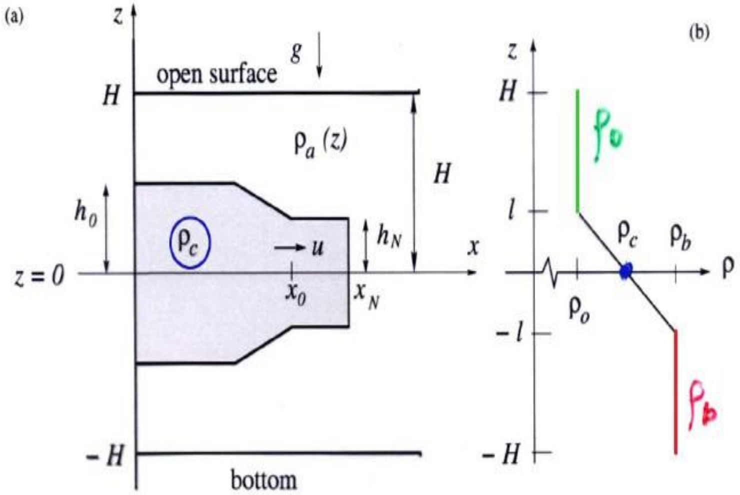
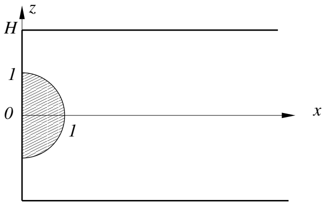


Figure 1: Schematic description of the system: (a) geometry after release from a rectangular lock; (b) density profile in the ambient (note $\rho_c = 0.5(\rho_b + \rho_o)$). In dimensionless form, horizontal and vertical lengths are scaled with x_0 and h_0, respectively. The subscripts denote: N - nose (or front); a - ambient;

Wu's "model"



Wu (1969) made experiments using cylinder lock, $H = 4$, full linear stratification with buoyancy frequency

$$\mathcal{N} = [(\rho_b/\rho_o - 1)g/H^*]^{1/2}$$

The curve-fitted data produced Wu's formula

$$\frac{x_N^*}{x_0} = \begin{cases} 1 + (0.29 \pm 0.04)(\mathcal{N} t^*)^{1.08 \pm 0.05} & (0 \leq \mathcal{N} t^* \leq 2.5) \quad \text{(I.C.S.)} \\ (1.03 \pm 0.05)(\mathcal{N} t^*)^{0.55 \pm 0.02} & (3 \leq \mathcal{N} t^* \leq 25) \quad \text{(P.C.S.)}, \end{cases}$$

I.C.S. means "initial collapse stage" and P.C.S. means "principal C. S ."

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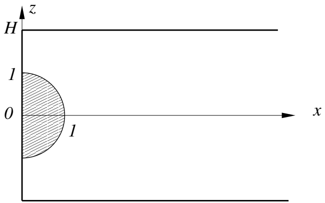

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I.C.S. means "initial collapse stage" and P.C.S. means "principal C. S ."

This formula was accepted as a general description of intrusions.

Kao (1976), Manins (1976), and Amen and Maxworthy (1980) tried to extend it, using experiments for rectangular locks and *adjustable constants*.

This was the accepted "theory". Simple BUT ....

Crisis

More experimental work was done by Faust and Plate (1984).

Faust and Plate (1984) summarized:

"intrusions into a linearly stratified environment behave very differently from theoretical calculations."

REASON:

Wu's formula is a curve-fit, not a physical "model." The word "theory" is used too loosely.

REMEDY:

Extend the SW approach to these problems. (Ungarish 2005).

Observation: the Boussinesq intrusion is composed of two mirror-image boundary currents. With some care, it is sufficient to solve the SW equations for the upper-half only. Analytical solutions exist for slumping and self-similar stages.

Note the times

1969-1984-2005

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Intrusions at mid-level of container, SW eqs

Use a one-layer approximation. Hydrostatic balances in z direction yield

$$\frac{\partial p_c}{\partial x} = \rho_o g' \frac{\partial h}{\partial x} [1 - \sigma(h)] .$$

$$\rho_c = \rho_o (1 + \varepsilon)$$

$$\rho_a = \rho_o [1 + \varepsilon \sigma(z)] ,$$

Scale the dimensional variables (denoted here by asterisks) as follows

$$\{x^*, z^*, h^*, l^*, H^*, t^*, u^*, p^*\} = \{x_0 x, h_0 z, h_0 h, h_0 l, h_0 H, Tt, Uu, \rho_o U^2 p\},$$

Nose condition :

$$u_N = Fr h_N^{1/2} \times [1 - \Lambda(h_N)]^{1/2} \mathcal{A} ,$$

where

$$\Lambda(h_N) = \frac{1}{h_N} \int_0^{h_N} \sigma(z) dz .$$

where

$$U = \left[\frac{\rho_c - \rho_a(z=1)}{\rho_o} h_0 g \right]^{1/2} = (h_0 g')^{1/2} \frac{1}{\mathcal{A}}, \qquad T = \frac{x_0}{U},$$

and

$$\mathcal{A} = [1 - \sigma(1)]^{-1/2} = \begin{cases} 1 & (l \le 1) \\ \sqrt{l} & (l > 1) \end{cases}$$

The resulting SW equations are

cont.:
x mom.:

$$\begin{bmatrix} h_t \\ u_t \end{bmatrix} + \begin{bmatrix} u & h \\ \mathcal{A}^2[1 - \sigma(h)] & u \end{bmatrix} \begin{bmatrix} h_x \\ u_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

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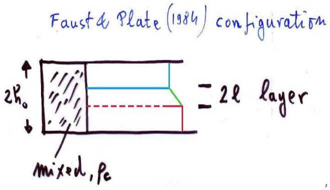
Does this resolve the dilemma of Faust and Plate ?

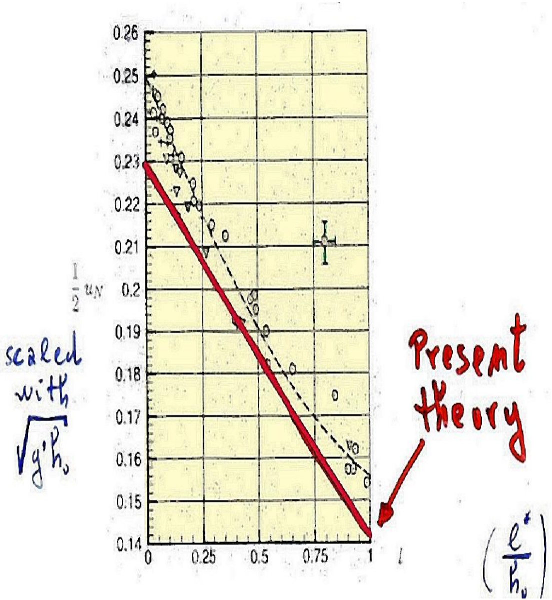


SW theory predicts:

constant u_N

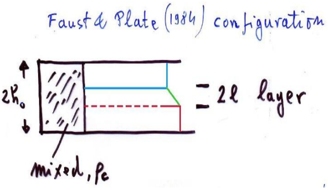
Free parameter: l/h_0

Agrees with experiment.



Very good agreement for the whole range.

Does this resolve the dilemma of Faust and Plate ?



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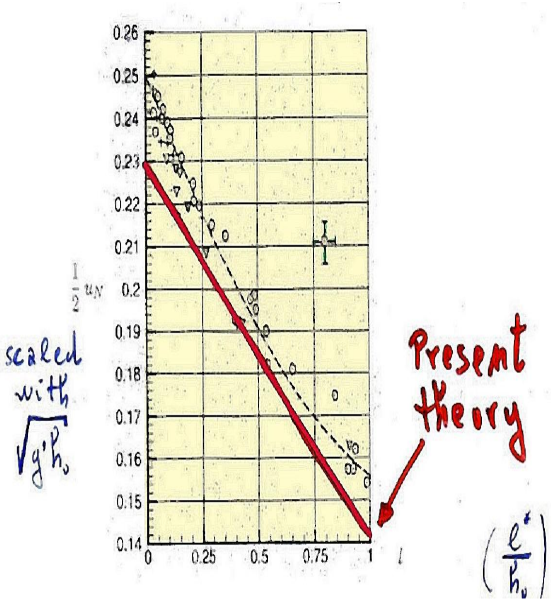

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SW, full depth stratification, slumping

The stratified fluid supports waves of max. speed $\mathscr{N} H^* / \pi$.

Question: is the intrusion faster or slower?

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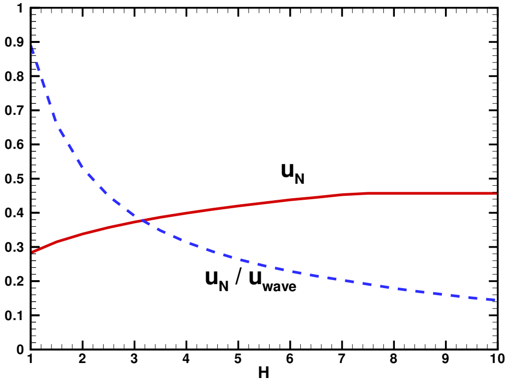
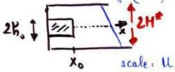
SW model analytical results for rectangular lock. u_N scaled with $\mathscr{N} h_0$.



The propagation is always slower than the wave (sub-critical).

SW, Navier-Stokes and experiments, $H = 1$

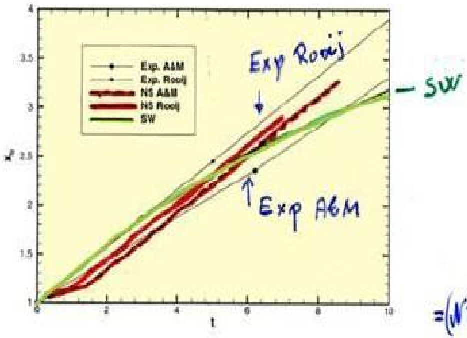
Amen & Maxworthy (1980), De Rooij (1999) configuration



$H = \frac{2H^*}{2h_0}$

scale: $U = N h_0$

Analytical result: constant initial "slumping" velocity, $u_N(H)$

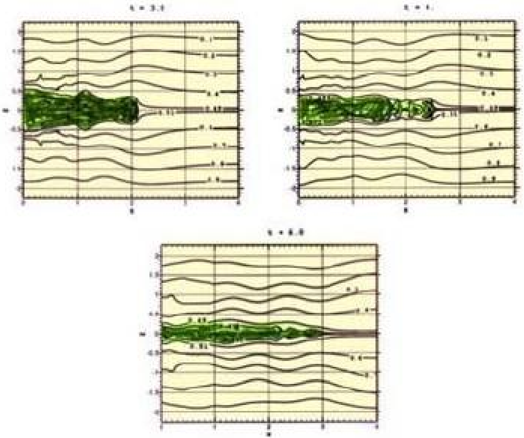


$= (N t^*)\left(\frac{h_0}{x_0}\right)$

Figure 9: Distance of propagation as a function of time for configurations with $H = 1$: results of experiments of Amen and Maxworthy, denoted A & M (with $h_0/x_0 = 1$) and of Rooij (with $h_0/x_0 = 0.5$); the corresponding NS simulations; and SW model.

$$U = N h_0 \ ; \ T = \frac{x_0}{U} \ ; \ \left(N^2 = g \frac{\Delta\rho}{\Delta z}\right)$$

Navier-Stokes, $H = 2.27$, isopycnals, $h_0/x_0 = 0.33$



Note the wave-head interaction in second and third frames, $t = 4$ and 6.

But the upstream perturbation is ≈ 0.

Simulates Run 111 of Amen-Maxworthy.

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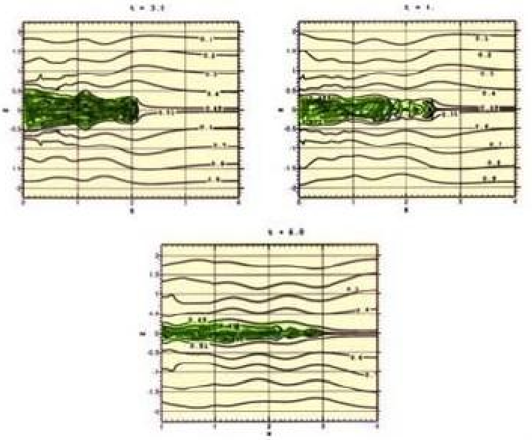

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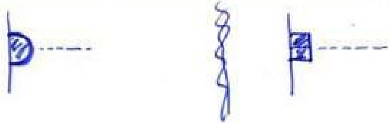
SW model proves: Wu's behavior is not universal !

Difference between cylindrical and rectangular locks



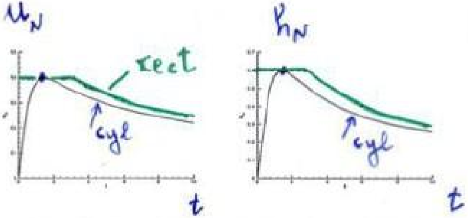


Figure 6: SW results for $H = 4$: Wu's cylindrical lock configuration (solid line) and rectangular lock counterpart (dashed line).

Note: max u_N and h_N are the same in both cases

Similarity solution for 2D intrusion

The SW equations for $S = 1$ and constant Fr are satisfied by

$$x_N(t) = K(t+\gamma)^{1/2}; \quad u = \dot{x}_N(t) y; \quad h = (b^2 + y^2)^{1/2} \dot{x}_N(t),$$

$$\text{where} \quad y = x/x_N(t), \quad b^2 = \frac{2}{Fr^2} - 1, \quad K, \gamma \text{ constants}$$

and the upper dot means differentiation in time.

Note difference from the homogeneous ambient case

$$x_N \sim t^{2/3}, \quad h \sim (C + y^2)\dot{x}_N^2(t))$$

Conservation of volume gives

$K = 1.362$ for cylindrical lock and 1.537 for rectangular lock.

Conclusion: SW similarity prediction agrees well with Wu's correlation

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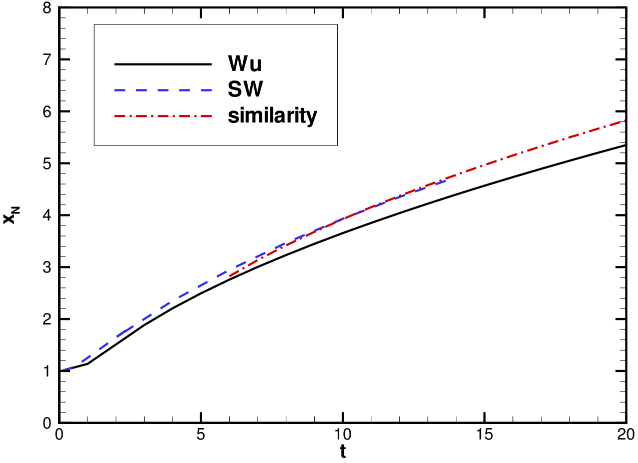
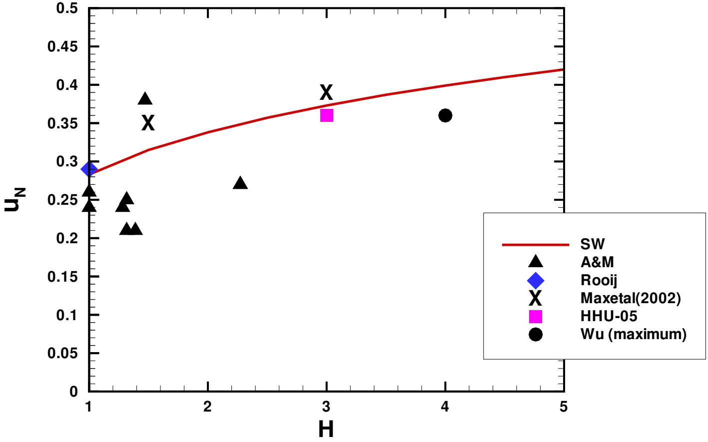
Wu's case: compare SW results to Wu's formula



The disagreement for "initial" and "principal" collapse stages is within the reported error bounds of Wu's formula.

Mid-level 2D intrusion in linear stratification

Slumping u_N (scaled with $\mathcal{N} h_0$) SW theory and experiments



The scatter suggests that the errors are in experiments, not in the theory.

Most experiments are old, and for small H.

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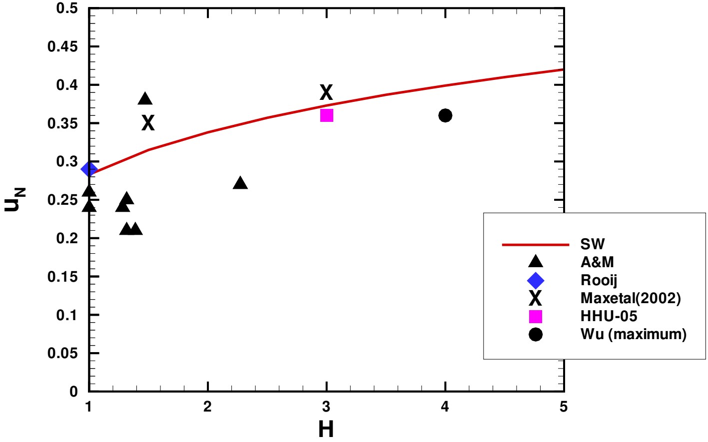

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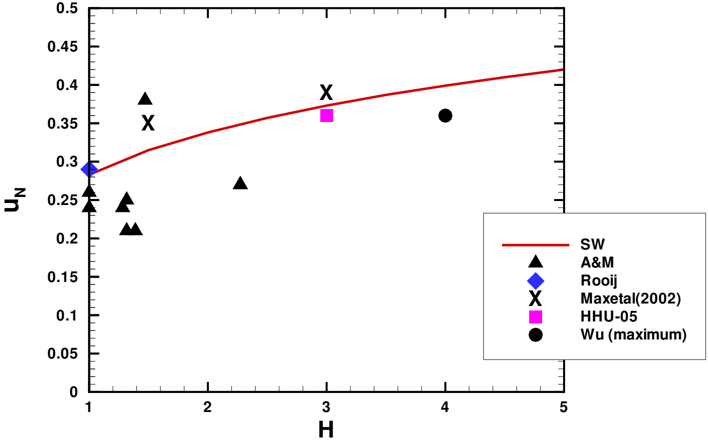

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Axisymmetric intrusion in linear stratification

The 2D formulation was extended to cylindrical motion in r, z coordinates.

A similarity solution was obtained (Ungarish and Zemach 2007).

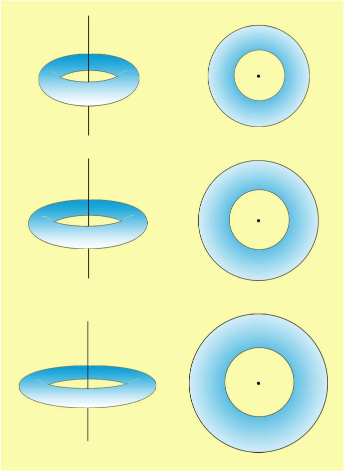
The propagation is $r_N = Kt^{1/3}$.

(In homogeneous case, $r_N = kt^{1/2}$).

Peculiar unexpected behavior:

the intruding fluid is in a ring (torus) of constant ratio of inner to outer radius.

The inner region is a “dry spot”.



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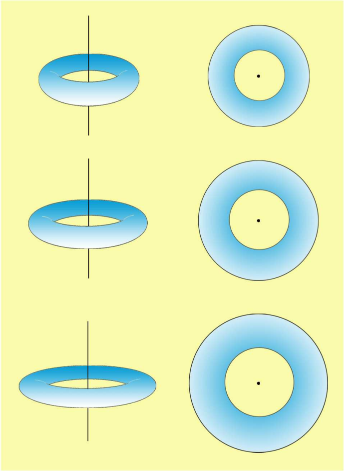

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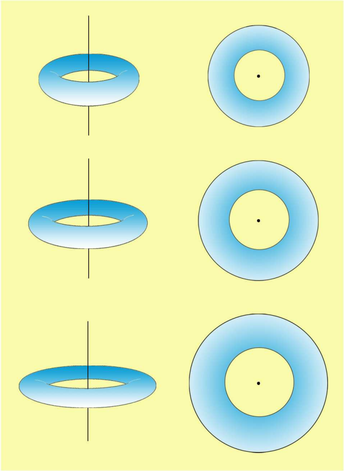

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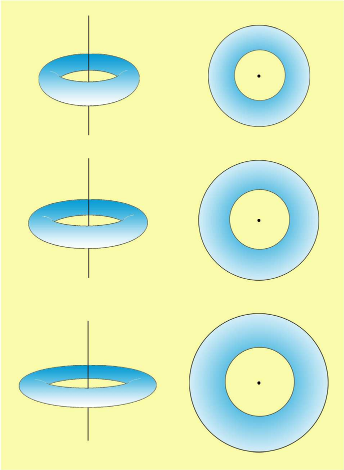
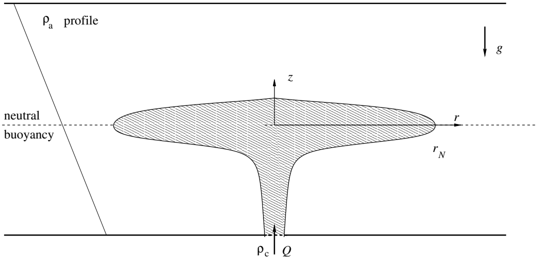


Axisymmetric intrusion with influx



Important problem:

Plume turns into intrusion.

Relevant to propagation of volcanic clouds (Suzuki and Koyaguchi 2009)

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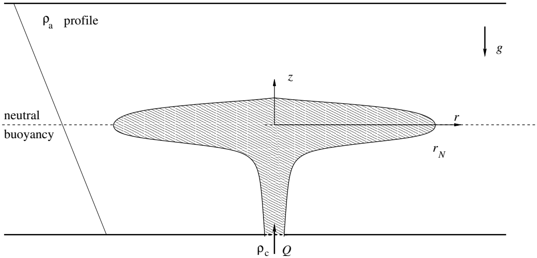

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Box-model: approximate the intrusion as a cylinder box of height $2h_N(t)$.

Volume conservation: $Qt = \pi r_N^2(t)[2h_N(t)]$;

Front condition $\frac{dr_N}{dt} = \frac{Fr}{\sqrt{2}}\mathcal{N} h_N$

Result: $r_N(t) = C(\mathcal{N}Q)^{1/3}t^{2/3}$.

Predicted $C = 0.59$.

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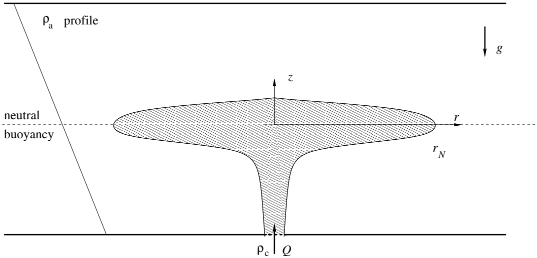

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Field measurements: spread with $t^{2/3}$, but $C \approx 0.40$.

The buoyancy frequency $\mathscr{N}$ is known in atmosphere and oceans. Q is a property of the phenomenon (i.e., volcanic eruption).

Conclusions

- The “thin layer” models provide useful and reliable information about the motion of gravity currents and intrusions.
- When properly scaled, the main propagation features can be reduced to simple equations which depend on a small number of dimensionless parameters. There still are open topics under research, e.g., the non-Boussinesq systems.
- The “models” work well when they are based on reliable physical mechanisms and are expressed in clear-cut balance equations with realistic initial conditions.
- “Extensions” of observations from one range of parameters (or geometry) to another may be misleading. One must be careful not to confuse between predictive governing equations and curve-fit equations. A good model is valid over a range of parameters, without adjustable constants.

Conclusions -continued

- It is actually amazing that complex physical flow-fields can be reduced to simple prediction equations. The reason is that the process occurs in some asymptotic range of the involved parameters. Many of the “complex” components are less important than observations and intuition suggest. The dominant governing balances are simple. The gravity current (intrusion) is an example of such a process.

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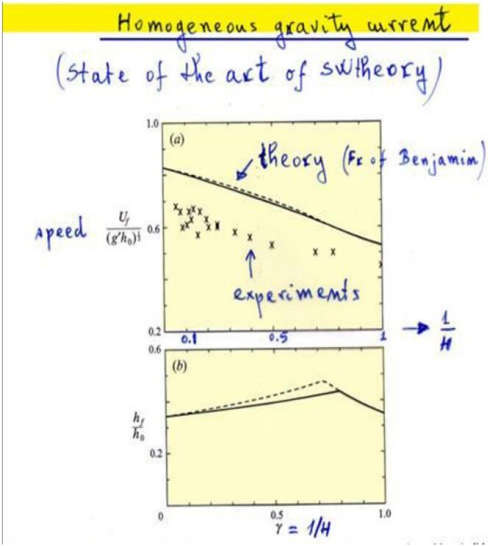
  The derivation of simple insightful models is one of the big challenges and benefits of the physical sciences. Let us hope that the computers will not make it redundant.

ACKNOWLEDGMENT

THANKS for the invitation ! ! !

State of the art $S = 0$, Klemp et al (1994)



Dissipation

In ideal conditions pressure on streamline satisfies Bernoulli's eq.

$$p_l^i(z) + \frac{1}{2}\rho_l(z)u_l^2(z) + \rho_l(z)\delta_l(z)g = p_r(z - \delta_l(z)) + \frac{1}{2}\rho_l(z)U^2 \quad (h \le z \le H)$$

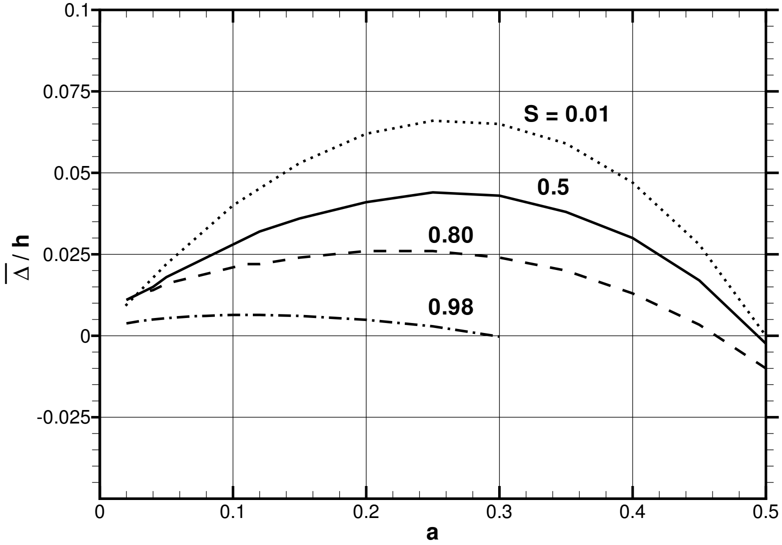
where the upperscript i denotes ideal energy-conserving flow. For the non-ideal flow, following Benjamin, we introduce the head loss on a streamline

$$\Delta(z) = \left[p_l^i(z) - p_l(z)\right]/(\rho_o g') \quad (h \le z \le H).$$

Define the average head loss

$$\overline{\Delta} = \frac{1}{H-h}\int_h^H \Delta(z)dz.$$

Steady current dissipation



the flow-force balance eq.

After algebra and reductions, we can express the flow-force balance as follows. Let

$$\gamma = (1-a)\sqrt{\frac{S}{a}\frac{1}{\hat{U}}}$$

$$f(\gamma) = 1-a+a(2-a)\gamma\cot\gamma+(a\gamma\cot\gamma)^2+\gamma^2\frac{a}{1-a}\left[(2-a)(1-\frac{1}{S})-\frac{1}{3}a^2\right] = 0. \quad (3)$$

The root(s) of this equation, for given a and S, provide the desired solution $Fr(a,S) = \hat{U} = (1-a)(S/a)^{1/2}/\gamma$.

Max et al (2002) Table

Appendix A

The experimental parameters are given in table 1.

Expt	h/H	$AR = L/h$	ρ_b	ρ_C	ρ_0	N	N_C	R	Fr	NT_x	X_x/h
1	2/3	2	1.032	1.035	1.004	1.351	1.421	1.107	0.255	24	9.18
2	2/3	2	1.033	1.045	1.004	1.374	1.634	1.414	0.375	33.7	18.96
3	2/3	2	1.035	1.090	1.005	1.397	2.352	2.833	0.637	18.7	17.87
4	1/3	4	1.044	1.070	1.005	1.593	2.057	1.667	0.317	16.1	15.31
5	1/3	4	1.037	1.119	1.003	1.489	2.750	3.412	0.565	9.2	15.59
6	1/3	4	1.034	1.075	1.004	1.398	2.151	2.367	0.438	12.4	16.29
7	1/3	4	1.034	1.065	1.004	1.398	1.993	2.033	0.375	12.9	14.51
8	1/3	4	1.033	1.037	1.005	1.350	1.443	1.143	0.182	12.7	6.93
9	1/3	4	1.033	1.049	1.005	1.350	1.702	1.589	0.290	18	15.66
10	1/3	4	1.036	1.045	1.003	1.467	1.655	1.273	0.232	17.6	12.25
11	1/3	4	1.034	1.034	1.003	1.422	1.422	1.000	0.131	9.9	3.89
12	1/3	4	1.035	1.048	1.004	1.421	1.703	1.435	0.269	17.1	13.80
13	2/3	4	1.064	1.099	1.008	1.907	2.433	1.629	0.437	16.6	10.88
14	2/3	2	1.065	1.139	1.008	1.932	2.917	2.280	0.555	15.5	12.90
15	1/3	4	1.065	1.094	1.006	1.958	2.390	1.490	0.287	19	16.36
16	1/3	4	1.064	1.072	1.006	1.942	2.080	1.147	0.190	15.5	8.84
17	1/3	4	1.067	1.112	1.005	2.001	2.638	1.738	0.327	19.6	19.23
18	1/3	4	1.066	1.163	1.008	1.945	3.170	2.655	0.463	13.3	18.47
19	1/3	4	1.065	1.088	1.007	1.941	2.294	1.397	0.264	19.6	15.52
20	1/3	4	1.065	1.122	1.006	1.958	2.746	1.966	0.379	15.3	17.40
21	1/3	4	1.067	1.082	1.006	2.000	2.235	1.249	0.233	10.4	7.27
22	1/3	4	1.067	1.079	1.006	2.000	2.181	1.189	0.210	14.2	8.95
23	1/3	8	1.068	1.085	1.008	1.965	2.232	1.291	0.249	29	21.66
24	1/2	8/3	1.067	1.069	1.008	1.948	1.983	1.036	0.211	18.3	7.72
25	1/2	8/3	1.066	1.082	1.008	1.940	2.202	1.288	0.295	28.9	17.05
26	1/2	8/3	1.067	1.115	1.008	1.957	2.639	1.819	0.415	20.6	17.10
27	1/2	8/3	1.069	1.181	1.007	2.007	3.363	2.808	0.588	20.1	23.64
28	1/2	8/3	1.035	1.042	1.004	1.410	1.582	1.259	0.291	28.2	16.41
29	1/2	8/3	1.035	1.040	1.003	1.444	1.555	1.159	0.267	28.1	15.01
30	1/2	16/3	1.036	1.054	1.003	1.456	1.827	1.575	0.382	18.3	13.98
31	1/2	8/3	1.068	1.139	1.007	1.990	2.927	2.162	0.483	15.3	14.78
32	1	4/3	1.068	1.072	1.009	1.947	2.019	1.075	0.295	60	17.70
33	1	8/3	1.068	1.104	1.007	1.990	2.505	1.584	0.448	29	12.99
34	1	4/3	1.034	1.072	1.004	1.386	2.103	2.302	0.597	21.5	12.84
35	1	4/3	1.034	1.094	1.004	1.398	2.421	3.000	0.701		
36	2/3	2	1.067	1.067	1.0075	1.965	1.965	1.000	0.230		

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