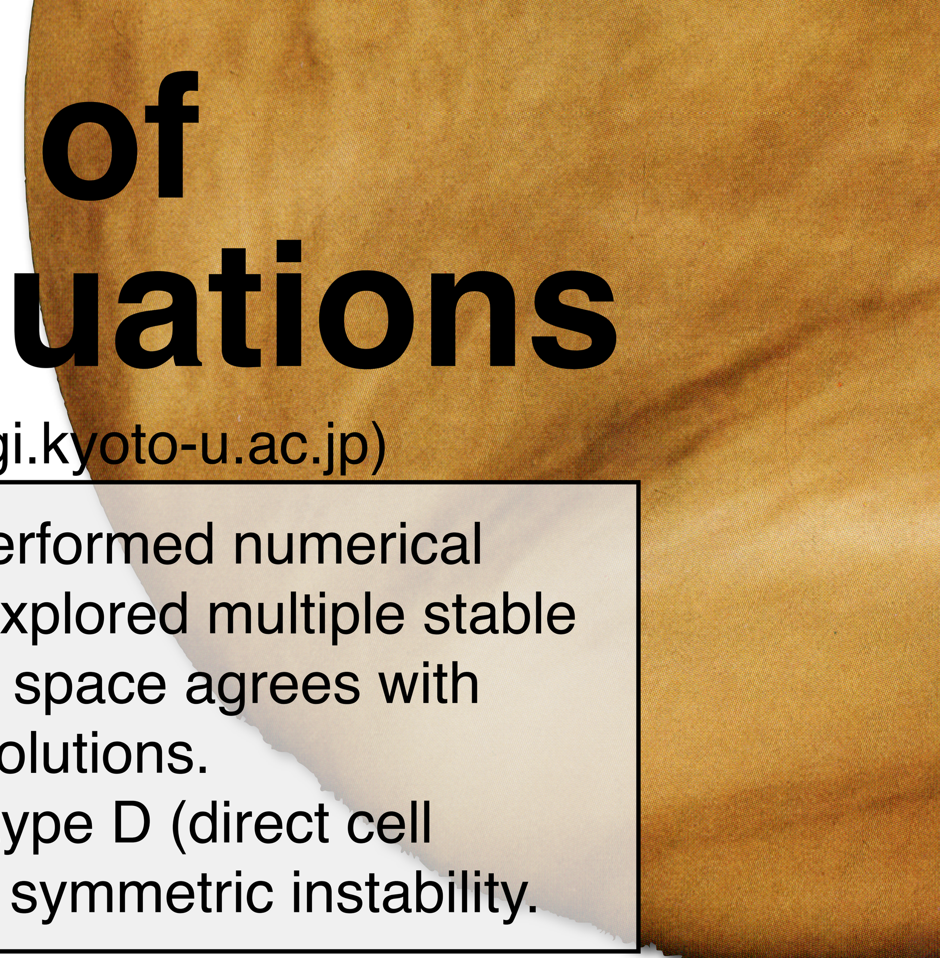

Multiple Stable Solutions of Boussinesq Fluid Primitive Equations

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Summary: To study multiple stable states of general circulations of planetary atmospheres, we performed numerical calculations of planetary axial symmetric 2-D Boussinesq fluid primitive equations minutely, and explored multiple stable solutions. The obtained position of the region of multiple solutions in non-dimensional parametric space agrees with Matsuda (1980, 1982) who used a low order model and suggested the existence of the multiple solutions.
Our numerical solutions show the characteristics of type V (Venusian thermal wind balance) and type D (direct cell balance) very well. There are some unsteady stable solutions, whose unsteadiness is caused by symmetric instability.

1. Introduction

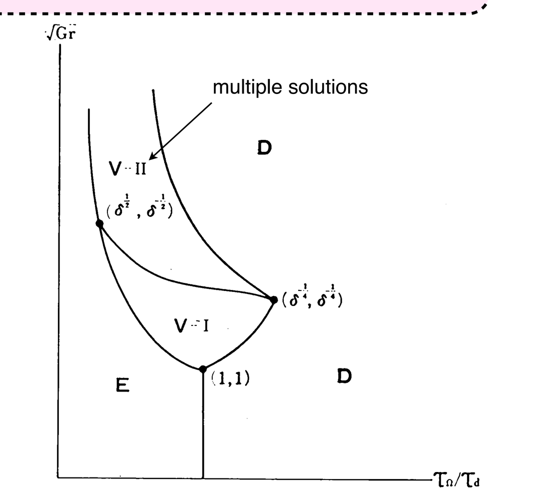
It is observed that the Venusian atmosphere rotates about 60 times faster than its solid planet. This phenomena is called super-rotation, and its mechanism is a hot topic of research of planetary atmospheres.

Meanwhile, it is suggested that another state of the atmospheric motion is possible in the Venus by Matsuda (1980, 1982). Using a planetary axial symmetric 2-D low order (truncated at wave number 3) idealized model, he showed that there are multiple equilibrium solutions:

type V (Venusian thermal wind balance):
a solution with *strong zonal wind* and *weak meridional circulation*, and
type D (direct cell balance):
a solution with *weak zonal wind* and *strong meridional circulation*

in some Venus-like parametric range (right figure). Recently Kido & Wakata (2008) succeeded to show multiple stable solutions in a Venus-like atmospheric general circulation 3-D model.

In this study, we explore multiple stable solutions, and investigate their properties in a wide parametric range in an axisymmetric 2-D Boussinesq fluid primitive system. This system is same as Matsuda's model, but we use a full non-linear high order model to obtain numerical solutions.



Matsuda's regiem diagram. V-II corresponds to the region of multiple solutions. (Matsuda, 1980)

2. Governing equations and boundary conditions

The axisymmetric 2-D Boussinesq fluid primitive equations are as follows,

Momentum equations

$$\frac{v}{a}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} - \frac{uv\tan\phi}{a} - 2\Omega v\sin\phi = \nu_H D_H(u) + \nu_V\frac{\partial^2 u}{\partial z^2},$$

$$\frac{v}{a}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + \frac{u^2\tan\phi}{a} + 2\Omega u\sin\phi = -\frac{1}{a}\frac{\partial \Phi}{\partial \phi} + \nu_H D_H(v) + \nu_V\frac{\partial^2 v}{\partial z^2},$$

Hydrostatic equation

$$\frac{\partial \Phi}{\partial z} = g\alpha\Theta,$$

Continuity equation

$$\frac{1}{a\cos\phi}\frac{\partial}{\partial \phi}(v\cos\phi) + \frac{\partial w}{\partial z} = 0.$$

Thermodynamic equation

$$\frac{v}{a}\frac{\partial \Theta}{\partial \phi} + w\frac{\partial \Theta}{\partial z} = -\frac{\Theta - \Theta_e}{\tau} + \kappa_V\frac{\partial^2 \Theta}{\partial z^2},$$

Basic potential temperature for Newtonian heating/cooling

$$\frac{\Theta_e}{\Theta_0} = 1 - \frac{2}{3}\Delta_H P_2(\sin\phi), \quad P_2(\sin\phi) = \frac{1}{2}(3\sin^2\phi - 1),$$

Horizontal diffusion (Becker, 2001)

$$D_H(u) = \frac{1}{a^2\cos\phi}\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial u}{\partial \phi}\right) - \frac{u}{a^2\cos^2\phi} + \frac{2u}{a^2},$$

$$D_H(v) = \frac{1}{a^2\cos\phi}\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial v}{\partial \phi}\right) - \frac{v}{a^2\cos^2\phi} + \frac{1}{a}\frac{\partial}{\partial \phi}\left[\frac{1}{a\cos\phi}\frac{\partial}{\partial \phi}(v\cos\phi)\right] + \frac{2v}{a^2}.$$

Boundary conditions

$$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial \Theta}{\partial z} = 0 \quad \text{at} \quad z = H,$$

$$u = v = w = \frac{\partial \Theta}{\partial z} = 0 \quad \text{at} \quad z = 0.$$

Symbols

(u, v, w): zonal, meridional, and vertical components of the velocity,
Θ: potential temperature, $\Phi = p/\varrho$, p: pressure, ϱ: density, ϕ: latitude, z: height,
a and Ω: radius and angular velocity of the planet, g: gravitational acceleration,
τ: time constant for Newtonian heating/cooling, $\varkappa_V$: vertical thermal diffusion coefficient,
ν_H and ν_V: horizontal and vertical momentum diffusion coefficient,
Θ_0: global mean of Θ_e, $\alpha = 1/\Theta_0$, Δ_H: fractional change of Θ_e from equator to pole.

3. Numerical experiments

To explore multiple solutions, parameter sweep experiments were carried out.

Swept non-dimensional parameters are $R_T \equiv \frac{gH\Delta_H}{a^2\Omega^2}$, $E_H \equiv \frac{\nu_H}{a^2\Omega}$, and $\frac{1}{\tau\Omega}$.

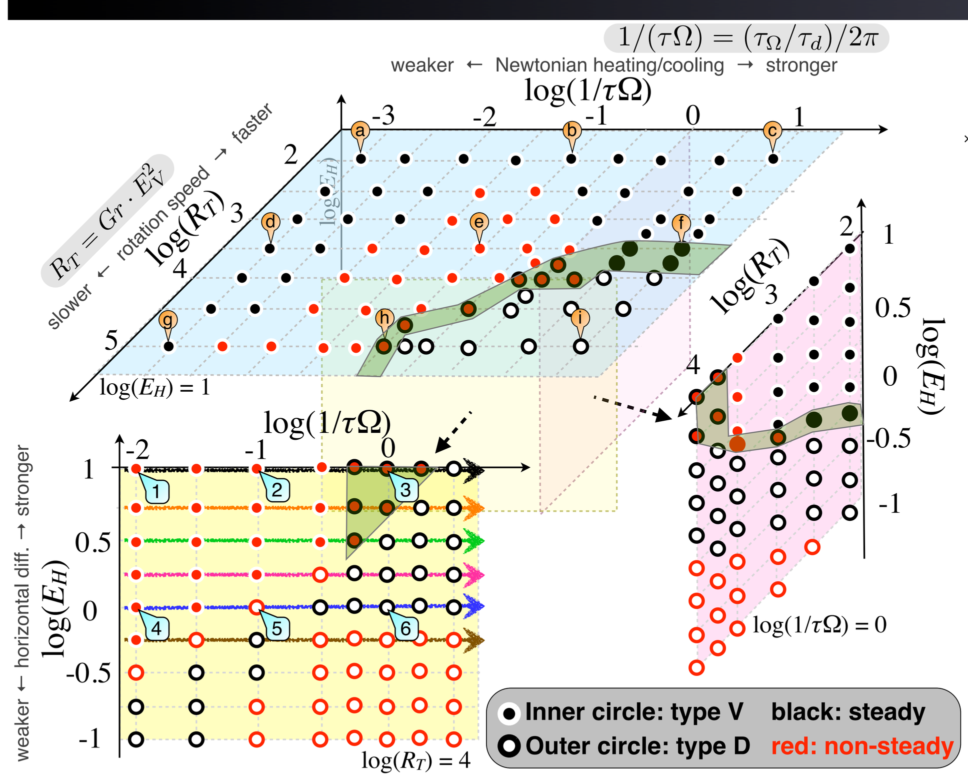
Fixed non-dimensional parameters are $Pr = \nu_V/\varkappa_V = 1$, $E_V = \nu_V/(H^2\Omega) = 10^{-2}$, and $\Delta_H = 1/10$. Dimensional constants are $\Theta_0 = 450$ K, $a = 6\times10^6$ m, $H = 5\times10^4$ m, and $g = 8.9$ m/s^2.

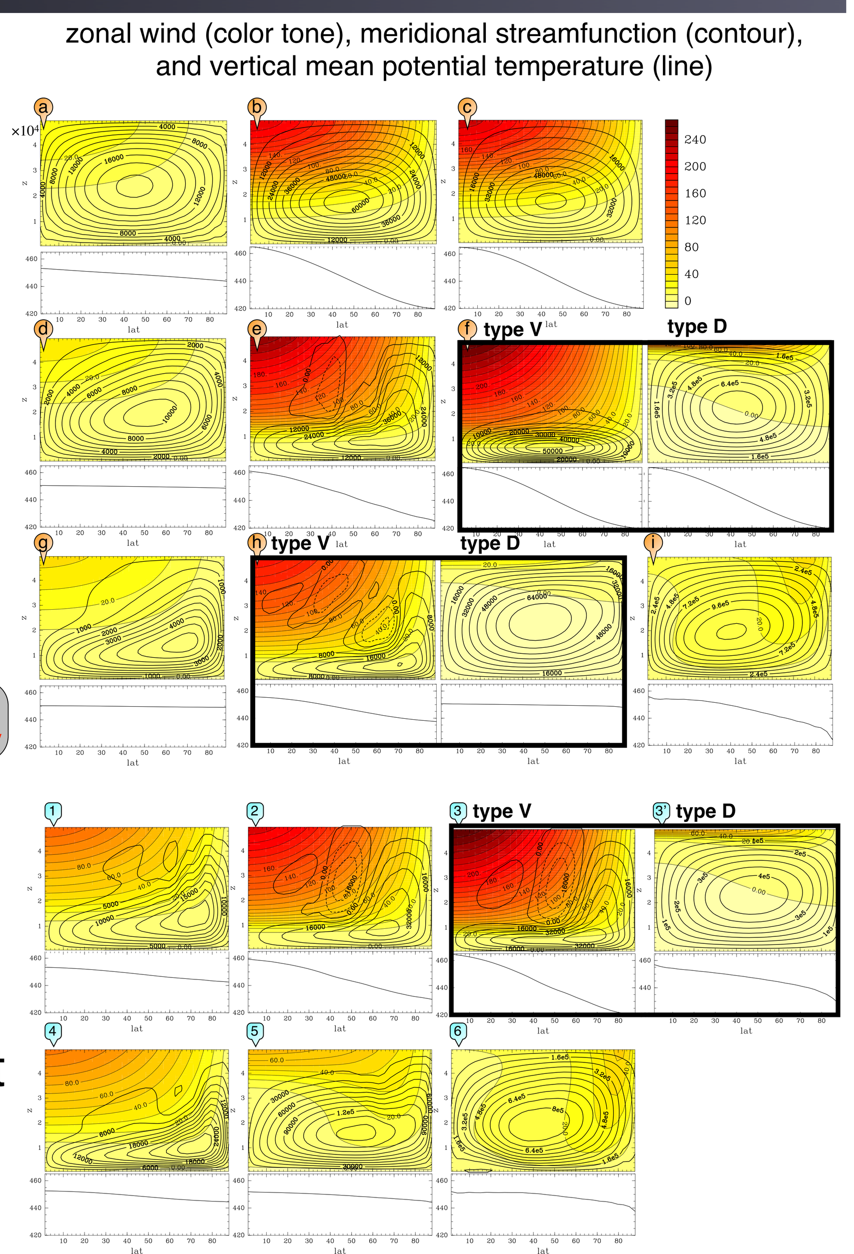
Numerical solutions were obtain by integrating a numerical model of the time-dependent version of the governing equations. The numerical model uses a spectral trasform method for meridional direction (truncation wave number 42; 32 grid point from equator to pole), a central difference method for vertical (50 layers), and the 4th order Runge-Kutta method for time-integrations.

We used a state at rest with a constant potential temperature and/or the stable solution achieved with other parametric values to obtain multiple solutions.

This model is almost same as the model used in Yamamoto et al. (2009) who explore the connection between the Held & Hou (1980) model of the Hadley circulation and Gierasch (1975)-Matsuda model of the super-rotation.

4. Numerical results

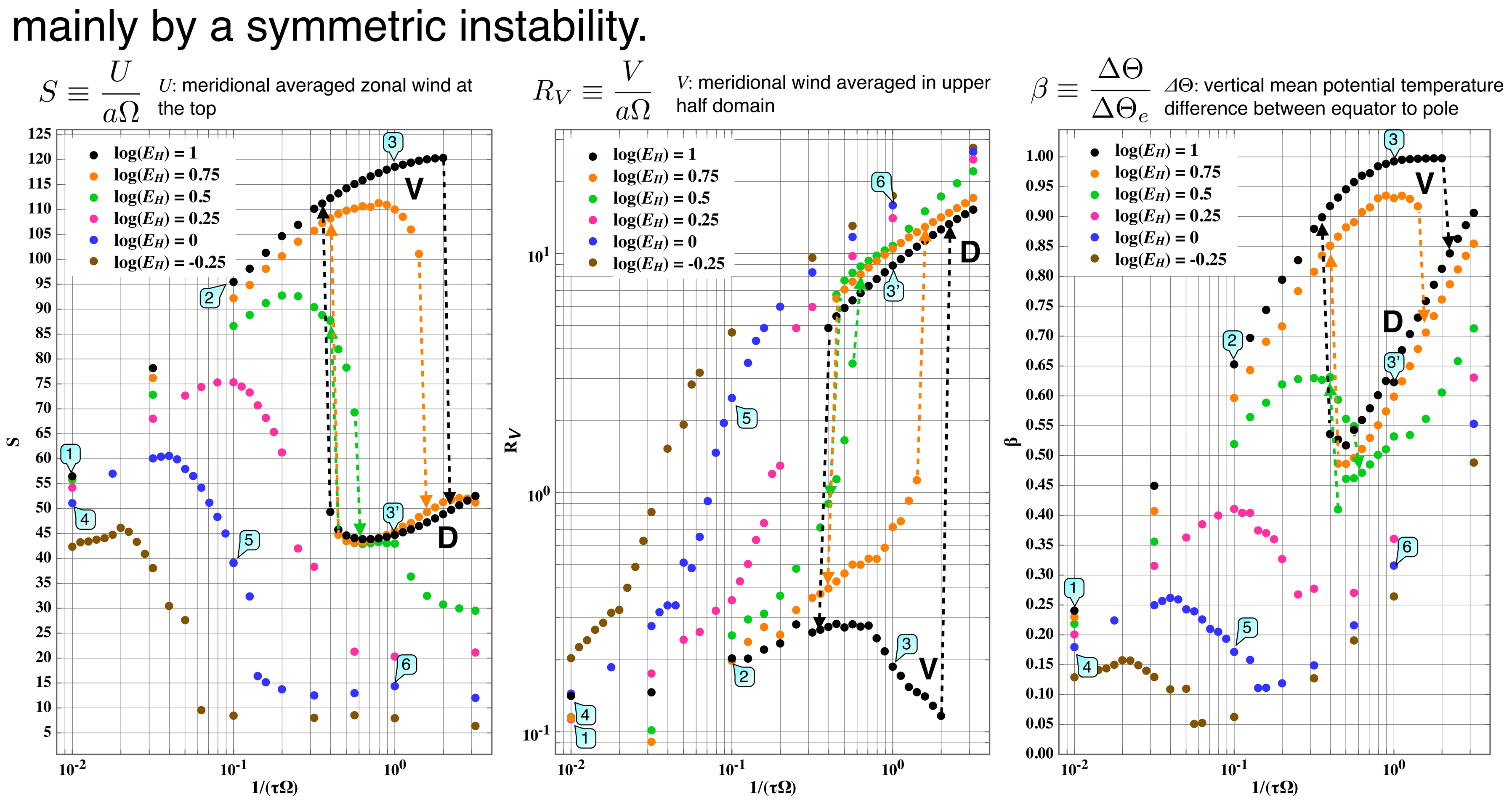




Left figure shows types of solution in the parametric space ($(\tau\Omega)^{-1}$, E_H, R_T). There is a region where both solutions type V and type D exist (green region).

Right figure shows numerical solutions at the steady states (time averaged states for non-steady stable solutions) at the positions shown by balloons.

We can see that obtained multiple solutions (balloons f, h, and #3) show characteristics of type V and type D, respectively. We obtain some unsteady stable solutions (balloons e, h, and #1-5). The unsteadiness is caused mainly by a symmetric instability.



The solutions in $(\tau\Omega)^{-1}$-E_H plane (yellow, $R_T = 10^4$), show that the region of multiple solutions disappears for $E_H \leqq 10^{0.25}$; this agrees with the results of Matsuda. A detail of this property is shown in the figures above, which show the values of S, R_V, and β, respectively (their definitions are shown at the top of each figure). We can see that the upper limit of $1/(\tau\Omega)$ for multiple solutions increases with E_H, but the lower limit is confined. The solutions located on the down slope in the figures of S and β (ballon #5) show intermediate properties between type V and type D.

5. Discussion

The location of the region of multiple solutions of our results agrees with the Matsuda's diagram qualitatively. However we should note that his diagram is based on the assumption of $\tau \sim H^2/\nu_V$.

We also computed a 3-D model without the high horizontal diffusion, however we did not obtain multiple solutions. This is because, an non-axisymmetric eddy momentum transport is not large enough to maintain a solution of type V.

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